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Abstract
We show that classical electrodynamics of massless charged particles and the Yang–Mills theory of massless quarks do not experience rearranging their initial degrees of freedom into dressed particles and radiation. Massless particles do not radiate. We propose a conformally invariant version of the direct interparticle action theory for these systems.

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1. Introduction

Rearrangement of the initial degrees of freedom appearing in the Lagrangian is a salient manifestation of self-interaction in field theory. The term 'rearrangement' was coined by Umezawa [1] who looked at spontaneous symmetry breaking for the presentation of advantages of this concept. The mechanism for rearranging classical gauge fields was further studied in [2–5]. What is the essence of this mechanism? While having unlimited freedom in choosing dynamical variables for describing a given field system, preference is normally given to those variables which are best suited for implementing fundamental symmetries. However, some degrees of freedom so introduced, if not all, are dynamically unstable. This gives rise to assembling the initial degrees of freedom into new, stable modes. For example, the Lagrangian of quantum chromodynamics is expressed in terms of quarks and gluons. If a system with these degrees of freedom would exhibit open color, there appears to be no reason for maintaining this system stable. Quarks and gluons combine in color–neutral clusters, hadrons and glueballs, in the cold phase, or else they form a lump of color–neutral quark–gluon plasma (QGP) in the hot phase. One further example is the Maxwell–Lorentz theory which is initially formulated in terms of mechanical variables $z^\mu(s)$ describing world lines of bare charged particles and the electromagnetic vector potential $A^\mu(x)$. The retarded interaction between these degrees of freedom makes them unstable, causing their rearranging into new dynamical entities: *dressed* charged particles and *radiation* [5].

There are exceptional systems. Their initial degrees of freedom remain unchanged under switching-on the interaction. Our interest here is with two theories of this kind: classical electrodynamics of massless charged particles and the Yang–Mills–Wong theory of massless colored particles. These theories have one property in common, *conformal invariance*. Owing to this symmetry, self-interaction does not create the renormalization of mass.

Conventional wisdom says that an accelerated charge emits radiation. However, we will see below that the net effect of radiation for a massless charged particle is compensated by an appropriate reparametrization of the world line. In other words, both radiation and dressing are absent from this theory. Classical electrodynamics of massless charged particles do not experience rearranging. It will transpire in section 2 that classical electrodynamics of massless charged particles is not a *smooth limit* of classical electrodynamics of massive charged particles. Conformal invariance has a dramatic effect on the picture as a whole: if this symmetry is broken, as in electrodynamics of massive charged particles, self-interaction is different from that of conformally invariant systems[1].

This argument is with minor modifications translated into a system of massless colored particles governed by the Yang–Mills–Wong dynamics. It is conformal invariance which is responsible for the lack of the mass renormalization. Furthermore, with this symmetry, the dynamics is constrained to the Abelian regime: the Cartan subgroup of the gauge group accommodates all the retarded field configurations.

Leptons of zero mass do not appear to exist. Nevertheless, the interest in a point charge moving at the speed of light is sometimes expressed in the literature [7, 8]. On the other hand, it is conceivable that quarks in QGP reveal themselves as massless particles. If a lump of QGP is formed in a collision of heavy ions, such as an Au + Au collision in the Relativistic Heavy Ion Collider (RHIC) at Brookhaven, then deconfinement triggers the chiral symmetry-restoring phase transition, whereby quarks become massless. As the data from RHIC measurements suggest (for a review see, e.g., [9]), the equation of state for QGP (pressure as a function of the energy density) above the transition temperature $T_{\rm c} \sim 160$ MeV is approximately $p = \frac{1}{3}\epsilon$, which is peculiar to a relativistic gas of massless particles[2]. The conformally invariant dynamics of massless particles discussed in this paper provides a laboratory for studying the properties of QGP.

The plan of the paper is as follows. In section 2, we briefly review the general properties of a conformally invariant classical system of massless charged particles in Minkowski spacetime $\mathbb{R}_{1,3}$. We point out that the principle of least action defies formulation for such systems. The reason for this is simple: in order that the Lagrangian be specified, a definite number of particles must be fixed. However, transformations of the conformal group $C(1, 3)$ convert a single world line into a two-branched world line, and hence do not preserve the number of particles. In section 3, we consider the retarded electromagnetic field $F_{\mu\nu}$ generated by a charge moving along a smooth lightlike world line. In section 4, we show that the radiation term of a massless charged particle drops out of the total energy–momentum balance equation. In section 5, classical electrodynamics of massless charged particles is recasted into a conformally invariant action-at-a-distance theory. The Yang–Mills–Wong theory of massless quarks is discussed in section 6. A central result of this section is that retarded solutions to the Yang–Mills equations with the source composed of massless quarks are Abelian. This is because these solutions are invariant under $C(1, 3)$. In section 7, we compare the Yang–Mills–Wong theories of massive and massless quarks. We then propose a path-integral description

[1] If conformal invariance is overlooked, as is the case in [6], then one can form the wrong impression of this system as that capable of the usual rearranging.

[2] In fact, for moderate temperatures $\sim T_{\rm c}$ accessible at RHIC, we are dealing with a strongly coupled perfect fluid rather than an ideal Stefan–Boltzmann gas. This is the most perfect fluid ever observed: the ratio of the QGP shear viscosity η to its entropy density s is about 0.1. For reference, liquid helium is characterized by $\eta/s \sim 10$.

of directly interacting massless colored particles. Some technical statements of sections 3 and 4 are justified in appendices A and B.

We adopt the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and follow the conventions of [5] throughout. The procedure of obtaining retarded solutions to classical gauge theories, without resort to Green's functions [3, 5] is represented at sufficient length to enable one to trace manipulations with singular distributions and have confidence in their validity.

2. Massless charged particles

Imagine a particle which is moving along a smooth null world line,

$$\dot z^2(\tau) = 0. \tag{1}$$

Here, z^μ stands for the world line parametrized by a monotonically increasing parameter τ. Derivatives with respect to τ are denoted by overdots. It follows from (1) that

$$\dot z \cdot \ddot z = 0. \tag{2}$$

Since $\dot z^\mu$ is lightlike, $\ddot z^\mu$ may be either spacelike or lightlike, aligned with $\dot z^\mu$. Let $\ddot z^2 < 0$. Then the trajectory is bent. As an example, we refer to a particle which orbits in a circle of radius r at an angular velocity of $1/r$. The history of this particle is depicted by a helical null world line of radius r wound around the time axis. The helix makes a close approach to the time axis as $r \to 0$. Note that, on a large scale, this particle traverses timelike intervals.

If $\ddot z^2 = 0$, then $\ddot z^\mu$ and $\dot z^\mu$ are parallel, and the trajectory is straight. Although we have nonzero components of $\ddot z^\mu$, the motion is uniform. Indeed, whatever the evolution parameter τ, the history is depicted by a straight null world line. We thus see that $\ddot z^\mu$ is, in this case, a fictitious acceleration. The occurrence of $\ddot z^\mu$ is an artifact of the choice of τ used for parametrizing the world line.

A massless particle of charge e is governed by

$$\varepsilon^\mu = \eta \ddot z^\mu + \dot\eta \dot z^\mu - e\dot z_\nu F^{\mu\nu}(z) = 0, \tag{3}$$

where η is an auxiliary dynamical variable, called einbein. Formally, equation (3) derives from the action[3]

$$S = -\int_{\tau'}^{\tau''} \mathrm{d}\tau \left(\frac{1}{2}\eta \dot z^2 + e\dot z \cdot A\right), \tag{5}$$

which was first proposed in [10]. Furthermore, varying η in (5) we come to (1).

The action (5) is reparametrization invariant if the transformation laws for η and z^μ are assumed to be, respectively, of the form

$$\delta\eta = \dot\epsilon \eta - \epsilon \dot\eta, \tag{6}$$

$$\delta z^\mu = \epsilon \dot z^\mu. \tag{7}$$

Here, ϵ is an infinitesimal reparametrization: $\delta\tau = \epsilon$. Under finite reparametrizations, $\tau \to \bar\tau$, the einbein transforms as

$$\eta \to \bar\eta = \frac{\mathrm{d}\bar\tau}{\mathrm{d}\tau}\eta. \tag{8}$$

[3] The kinetic term of the action (5) can be extended to include spin degrees of freedom. With real elements of a Grassmann algebra θ^μ and θ_5, the action for a free massless spinning particle reads

$$S = -\int_{\tau'}^{\tau''} \mathrm{d}\tau \left[\frac{1}{2}\eta \dot z^2 + \frac{\mathrm{i}}{2}(\dot\theta^\mu \theta_\mu + \dot\theta_5\theta_5) + \mathrm{i}\chi\theta^\mu \dot z_\mu\right] + \frac{\mathrm{i}}{2}[\theta^\mu(\tau')\theta_\mu(\tau'') + \theta_5(\tau')\theta_5(\tau'')], \tag{4}$$

where χ is a Grassmann-valued Lagrange multiplier (for details see, e.g., [11], and references therein). In addition to reparametrization symmetry, the action (4) is invariant under local (τ) and global (x^μ) supersymmetry transformations. However, we do not explore this supersymmetric extension in the present paper.

In view of this invariance, we are entitled to handle the reparametrization freedom making the dynamical equations as simple as possible. In particular, for some choice of the evolution parameter τ, the einbein can be converted to a constant, $\eta = \eta_0$, and (3) becomes

$$\eta_0 \ddot{z}^\mu = e \dot{z}_\nu F^{\mu\nu}(z). \tag{9}$$

Consider a system of N massless charged particles. They generate the electromagnetic field $F^{\mu\nu}$ according to Maxwell's equations,

$$\mathcal{E}^{\lambda\mu\nu} = \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0, \tag{10}$$

$$\mathcal{E}^\mu = \partial_\nu F^{\mu\nu} + 4\pi j^\mu = 0, \tag{11}$$

$$j^\mu(x) = \sum_{I=1}^N e_I \int_{-\infty}^{\infty} \mathrm{d}\tau_I\, \dot{z}_I^\mu(\tau_I)\delta^4[x - z_I(\tau_I)]. \tag{12}$$

Equations (1), (3) and (10)–(12) are basic for the subsequent analysis. Joint solutions to these equations will in principle tell us all we need to know about the behavior of this closed system of N massless charged particles and electromagnetic field.

At first glance it would seem that the whole dynamics is encoded in the action

$$S = -\sum_{I=1}^N \frac{1}{2}\int_{\tau'}^{\tau''} \mathrm{d}\tau_I\, \eta_I \dot{z}_I^2 - \int \mathrm{d}^4x \left(j_\mu A^\mu + \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} \right). \tag{13}$$

Indeed, varying η_I, z_I^μ and A^μ gives (1), (3) and (11). Furthermore, the stress–energy tensor associated with (13) is $T_{\mu\nu} = t_{\mu\nu} + \Theta_{\mu\nu}$, where

$$t_{\mu\nu}(x) = \sum_{I=1}^N \int_{-\infty}^{\infty} \mathrm{d}\tau_I\, \eta_I(\tau_I) \dot{z}^I_\mu(\tau_I)\dot{z}^I_\nu(\tau_I)\delta^4[x - z_I(\tau_I)], \tag{14}$$

$$\Theta_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu^\alpha F_{\alpha\nu} + \frac{\eta_{\mu\nu}}{4} F_{\alpha\beta}F^{\alpha\beta} \right). \tag{15}$$

Evidently

$$T_\mu^\mu = 0. \tag{16}$$

This implies invariance under the group of Weyl rescalings in four dimensions, and hence conformal invariance [12].

In fact, such is the case only when it is granted that spacetime is equipped with Euclidean signature $(++++)$. We then are entitled to regard equation (13) as the action which implies the dynamical equations (1), (3) and (11)–(12), and renders $T_{\mu\nu}$ traceless.

On the assumption of Lorentzian signature $(+---)$, the principle of least action for a particle moving along a null world line defies precise formulation. Consider a null curve with endpoints separated by a timelike interval. A transformation of the conformal group $C(1,3)$ can map these points so that their images have a spacelike separation. Hence, the conventional Lagrangian setting is not compatible with the requirement of conformal invariance. On the other hand, if the signature $(+---)$ is changed by $(++++)$, then the distinction between timelike and spacelike intervals disappears, and conformal symmetry, entrusted to the group $C(4)$, presents no special problem.

Let us suppose that the integration limits in (13) are extended from the remote past to the far future. The group $C(1,3)$ maps an infinite null curve to other infinite null curves. Rosen [13] suggested to consider transformations of $C(1,3)$ as leaving both spacetime and the coordinate system unaffected but serving to map only the world lines of charged particles and field configurations generated by these particles. With this interpretation in mind, we come to a remarkable result: any null curve, different from a straight line, can be transformed to a two-branched null curve. The transformed picture displays the presence of two particles, or, more precisely, a particle and an antiparticle (for more details see [5], section 5.3). Therefore, the number of particles is not preserved by $C(1,3)$.

The Lagrangian description assumes fixing a definite number of particles N. However, in this case we have a completely different situation. There is an infinite set of layouts with different N related to each other by conformal transformations. Every physically valid picture described by a simultaneous solution to Maxwell's equations (10)–(12) and the dynamical equations for massless charged particles (1) and (3) can be obtained via a transformation of the group $C(1,3)$ from a particular picture. We note that the effect of $C(1,3)$ on a given system is not the mere running over states of this system; conformal transformations may substitute this system with other feasible systems.

In contrast, given Euclidean signature, a single layout with a fixed number of particles turns out to be invariant under the conformal group. What is the reason for this drastic distinction between a theory invariant under $C(1,3)$ and its Euclidean conterpart invariant under $C(4)$? Let us take a closer look at a special conformal transformation

$$x_\mu \to x'_\mu = \frac{x_\mu - b_\mu x^2}{1 - 2b\cdot x + b^2x^2}. \tag{17}$$

The denominator can be rewritten as

$$1 - 2b\cdot x + b^2x^2 = b^2(x-a)^2, \qquad a_\mu = b_\mu/b^2. \tag{18}$$

In Minkowski spacetime $\mathbb{R}_{1,3}$, the mapping (17) is singular at the light cone $(x-a)^2 = 0$. Any null curve, different from a straight line, intersects this cone twice. One intersection point is mapped onto the remote past, while the other point is mapped onto the far future. This is another way of stating that the transformed curve is two branched. In Euclidean spacetime $\mathbb{R}_4$, the mapping (17) is singular at a single point $x_\mu = a_\mu$. If a curve does not pass through a_μ, none of the points on this curve is mapped onto infinity.

One further remark is in order. It is not sufficient to specify the Lagrangian if we are to define a classical theory. In addition, we fix geometry, boundary conditions, and a class of allowable functions which represents the space of dynamically realizable configurations. One can envision that some Lagrangian is well suited to a particular geometry (in the sense that the principle of least action is consistently formulated) and yet incompatible with a contiguous geometry. It is just the Lagrangian shown in (13) whose definition displays extreme sensitivity to switching between Euclidean and Lorentzian signatures. It seems appropriate to begin with the Lagrangian formulation in $\mathbb{R}_4$. Thereafter, we attempt at grafting the equations of motion and other dynamical structures onto $\mathbb{R}_{1,3}$ by an analytical continuation from $\mathbb{R}_4$.

For comparison purposes, we turn to the dynamics of a massive particle. The action for a particle of mass m interacting with an external electromagnetic field reads

$$S = -\int_{\tau'}^{\tau''} d\tau \left[\frac{1}{2}\left(\eta\dot{z}^2 + \frac{m^2}{\eta}\right) + e\dot{z}\cdot A\right]. \tag{19}$$

It is well known that the principle of least action for (19) is consistently formulated in both $\mathbb{R}_4$ and $\mathbb{R}_{1,3}$. Varying z^μ we again come to (3). However, variation of η results in

$$\dot{z}^2\eta^2 - m^2 = 0. \tag{20}$$

Since $\dot z^2 > 0$, one can readily solve this equation to show that η is fixed at the extremals: $\eta = m/\sqrt{\dot z \cdot \dot z}$. Combining (20) and (3) we arrive at the familiar equation of motion for a massive particle, parametrized by the proper time $\mathrm{d}s = \mathrm{d}\tau\sqrt{\dot z \cdot \dot z}$,

$$m\ddot z^\mu = e\dot z_\nu F^{\mu\nu}(z), \tag{21}$$

where the overdot denotes the derivative with respect to the proper time.

The action for a massive particle (19) becomes the action for a massless particle (5) as $m \to 0$, which, however, does not imply that the dynamics of a massive particle tends to the dynamics of a massless particle in this limit. The gist of the dissimilarity is in the sets of allowable world lines. For $m \neq 0$, the set $\mathfrak{S}$ of allowable world lines associated with the action (19) consists of smooth timelike curves, while, for $m = 0$, the corresponding set $\mathfrak{S}_0$ associated with the action (5) involves smooth null curves. Both $\mathfrak{S}$ and $\mathfrak{S}_0$ are invariant under $C(1,3)$ in the sense that the image of a timelike curve is timelike and the image of a null curve is null, but the sets $\mathfrak{S}$ and $\mathfrak{S}_0$ do not share common elements.

For $m \neq 0$,

$$t^\mu_\mu(x) = m\int_{-\infty}^{\infty} \mathrm{d}s\, \delta^4[x - z(s)], \tag{22}$$

and so

$$T^\mu_\mu \neq 0. \tag{23}$$

We thus see that if timelike curves are among the allowable world lines, then the theory does not enjoy the property of conformal invariance. This argument can be reversed. A conformally invariant theory is not a smooth limit of a related theory in which conformal invariance is broken; these theories are divorced just because their associated sets of allowable world lines, $\mathfrak{S}$ and $\mathfrak{S}_0$, do not overlap.

3. Electromagnetic field due to a massless charged particle

Consider a charge which is moving along a smooth null world line. We set $e_1 = e$, $e_2 = e_3 = \cdots = 0$ in (12), and look for an exact solution to Maxwell's equations (10)–(12) using the covariant retarded variable technique similar to that developed in [2–5]. We define the vector $R^\mu = x^\mu - z^\mu(\tau_{\mathrm{ret}})$ drawn from the point on the world line where the retarded signal was emitted, $z^\mu(\tau_{\mathrm{ret}})$, to the point x^μ where the signal was received. The constraint $R^2 = 0$ implies

$$\partial_\mu \tau = \frac{R_\mu}{R \cdot \dot z}. \tag{24}$$

From here on, we omit the subscript 'ret'.

The scalar

$$\rho = R \cdot \dot z \tag{25}$$

measures the separation between $z^\mu(\tau_{\mathrm{ret}})$ and x^μ. Indeed, let us choose a particular Lorentz frame in which

$$\dot z^\mu = (1, 0, 0, 1), \qquad R^\mu = r(1, \mathbf{n}) = r\,(1, \sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\,. \tag{26}$$

Here ϑ and φ are zenith and azimuth angles, respectively. From $R \cdot \dot z = r(1 - \cos\vartheta)$ it follows that ρ varies smoothly from 0 to ∞ as x^μ moves away from $z^\mu(\tau_{\mathrm{ret}})$, except for the case that R^μ points in the direction of $\dot z^\mu$. The variables $\tau, \rho, \vartheta, \varphi$ are an alternative to Cartesian coordinates. An obvious flaw of these coordinates is the presence of singular rays $\hat R_\mu$ aligned with the tangent vectors $\dot z_\mu$. Note that the surface swept out by the singular ray $\hat R_\mu$ is a two-dimensional warped manifold $\mathcal{M}_2$.

Combining (24) and (25) gives

$$\partial_\mu \tau = \frac{R_\mu}{\rho} = c_\mu. \tag{27}$$

Accordingly,

$$\partial_\mu \rho = \dot{z}_\mu + c_\mu[(R \cdot \ddot{z}) - \dot{z}^2]. \tag{28}$$

We retain the term $-c_\mu \dot{z}^2$ (which is identically zero unless the constraint $\dot{z}^2 = 0$ is ignored) in (28) for later use. Introducing one more retarded scalar

$$\lambda = (R \cdot \ddot{z}) - \dot{z}^2, \tag{29}$$

we come to the differentiation rule

$$\partial_\mu \rho = \dot{z}_\mu + \lambda c_\mu, \tag{30}$$

which is universally applicable to both timelike and null world lines.

Following the line of [2–5], the appropriate ansatz is given by

$$A^\mu = \dot{z}^\mu \Phi(\rho) + R^\mu \Psi(\rho). \tag{31}$$

We use the differentiation rules (27) and (30) to yield

$$F_{\mu\nu} = R_\mu U_\nu - R_\nu U_\mu, \tag{32}$$

$$\rho U_\nu = \ddot{z}_\nu \Phi + \dot{z}_\nu (\lambda \Phi' - \Psi - \rho \Psi') \tag{33}$$

and

$$\partial_\mu F^{\mu\nu} = 2U^\nu + (R \cdot \partial) U^\nu + \frac{\dot{z}^\nu}{\rho}(R \cdot U) - R^\nu (\partial \cdot U). \tag{34}$$

Since our main interest here is with the retarded solution to Maxwell's equations outside the world line, we equate to zero the coefficients of the three linearly independent vectors $\ddot{z}^\nu$, $\dot{z}^\nu$ and R^ν. In the latter case, we equate to zero separately the coefficients of $(R \cdot \dddot{z})$, $(R \cdot \ddot{z})^2$, and the sum of remaining terms of the coefficient of R^ν. We thus arrive at an overdetermined system of ordinary differential equations with Φ and Ψ as the unknown functions. Integrating this system is easy, so that we omit this part of mathematical treatment, and turn to the net result. The retarded solution to Maxwell's equations (10)–(11) is given by

$$A^\mu = q \frac{\dot{z}^\mu}{\rho}, \tag{35}$$

modulo gauge terms proportional to $R^\mu/\rho = \partial^\mu \tau$. Here, q is an integration constant.

Of course, Maxwell's equations (10)–(12) can be conveniently solved using Green's function method. Again, the retarded solution A^μ takes the form of equation (35), with q being equal to e. However, the above procedure will prove useful (in fact the only available) in looking for retarded solutions to the Yang–Mills equations.

The retarded electromagnetic field due to a charge moving along a null world line is

$$F_{\mu\nu} = F^{\rm r}_{\mu\nu} + F^{\rm ir}_{\mu\nu}. \tag{36}$$

The first term $F^{\rm r}_{\mu\nu}$ (r for regular) is

$$F^{\rm r}_{\mu\nu} = R_\mu V_\nu - R_\nu V_\mu, \tag{37}$$

$$V_\mu = \frac{q}{\rho^2}\left(-\dot{z}_\mu \frac{\ddot{z} \cdot R}{\rho} + \ddot{z}_\mu\right). \tag{38}$$

The second term $F^{\rm ir}_{\mu\nu}$ (ir for irregular) is

$$F^{\rm ir}_{\mu\nu} = q\,\frac{\dot z^2}{\rho^2}(c_\mu \dot z_\nu - c_\nu \dot z_\mu). \tag{39}$$

This term is everywhere zero except for the surface $\mathcal{M}_2$ formed by the singular rays $\hat R_\mu$.

The reader may wish to circumvent the procedure of integrating Maxwell's equations, but content himself (or herself) with verifying that the retarded field (36)–(39) is indeed the desired solution. This can be done through the use of the formulae

$$R\cdot V = 0, \tag{40}$$

$$\partial\cdot V = \frac{1}{\rho^2}\left[(\ddot z\cdot c) - \frac{1}{2}\frac{\rm d}{{\rm d}\tau}\right]\dot z^2, \tag{41}$$

and

$$(R\cdot\partial)\rho = \rho, \qquad (R\cdot\partial)V^\mu = -2V^\mu. \tag{42}$$

We note in passing the relations

$$\dot z\cdot V = -\frac{q}{\rho^2}\left[(\ddot z\cdot c) - \frac{1}{2}\frac{\rm d}{{\rm d}\tau}\right]\dot z^2, \tag{43}$$

$$V^2 = q^2\frac{\ddot z^2}{\rho^4}, \tag{44}$$

which will be useful in the subsequent discussion.

Let $\overline{\mathbb{M}}_4$ be all spacetime minus the singular manifold $\mathcal{M}_2$. For the regular part of the electromagnetic field $F^{\rm r}_{\mu\nu}$, both invariants $\mathcal{P} = \frac{1}{2}{}^*F_{\mu\nu}F^{\mu\nu}$ and $\mathcal{S} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ are vanishing in $\overline{\mathbb{M}}_4$. Therefore, a massless charged particle generates the retarded electromagnetic field whose regular part is a null field in $\overline{\mathbb{M}}_4$.

We determine the constant of integration q in (35), (38) and (39) by invoking Gauss' law. We first find the total flux of $\mathbf{E}^{\rm ir}$ through a sphere enclosing the charge e. We choose a Lorentz frame in which $\dot z^\mu$ and R^μ take the form of (26), and integrate $F^{\rm ir}_{0i}$ over a sphere $r = \ell$,

$$\int {\rm d}\mathbf{S}\cdot\mathbf{E}^{\rm ir} = e\dot z^2\int_0^{2\pi}{\rm d}\varphi\int_0^\pi {\rm d}\vartheta\,\frac{\sin\vartheta}{(\dot z_0 - |\dot{\mathbf{z}}|\cos\vartheta)^2} = 4\pi q. \tag{45}$$

It is shown in appendix A that this surface integral of $\mathbf{E}^{\rm r}$ is zero. Therefore, $q = e$.

We thus see that the total flux of $\mathbf{E}^{\rm ir}$, concentrated on the singular ray $\hat R_\mu$ which issues out of the charge e, is $4\pi e$. This resembles the Dirac picture of a magnetic monopole: the magnetic field $\mathbf{B}$ due to this monopole shrinks in a string. This string begins at the magnetic charge and goes to spatial infinity, so that the total flux of $\mathbf{B}$ flowing along the string equals the magnetic charge times the factor 4π.

It may be worth pointing out that the factor $\dot z^2$ disappears from equation (45) because it is canceled by the identical factor that arose in the denominator owing to the solid angle integration of ρ^{-2}. If we would have ρ^{-s} with s other than 2, then this mechanism would fall short of the required cancellation. In particular, a similar surface integral of the stress–energy tensor term quadratic in $F^{\rm ir}_{\mu\nu}$ is zero (see appendix B).

This consideration can be readily adapted to the advanced boundary condition. The advanced covariant technique can result from the retarded covariant technique if $\dot z$ is substituted for $-\dot z$ in every pertinent relation.

4. Massless charged particles do not emit radiation

We now look for a joint solution to the set of equations (10)–(12), (1) and (3). We begin with the Noether identity

$$\partial_\mu T^{\lambda\mu} = \frac{1}{8\pi}\mathcal{E}^{\lambda\mu\nu}F_{\mu\nu} + \frac{1}{4\pi}\mathcal{E}_\mu F^{\lambda\mu} + \int_{-\infty}^{\infty} \mathrm{d}\tau\, \varepsilon^\lambda(z)\delta^4[x - z(\tau)]. \tag{46}$$

Here, $T^{\mu\nu} = t_{\mu\nu} + \Theta_{\mu\nu}$ is the symmetric stress–energy tensor of this system, $t_{\mu\nu}$ and $\Theta_{\mu\nu}$ are given by (14) and (15), $\mathcal{E}^{\lambda\mu\nu}$, $\mathcal{E}_\mu$, and ε^λ are the left-hand sides of (10), (11) and (3), respectively. Were it not for divergences, the local conservation law for the stress–energy tensor $\partial_\mu T^{\lambda\mu} = 0$ would imply that both the equation of motion for bare particles $\varepsilon^\lambda = 0$ and the field equations $\mathcal{E}_\mu = 0$ and $\mathcal{E}^{\lambda\mu\nu} = 0$ simultaneously hold.

We first discuss the case that only a single particle is in the universe. Assume that $\mathcal{E}^{\lambda\mu\nu} = 0$ but $\mathcal{E}_\mu$ is nonzero. Then $F_{\mu\nu}$ may be regarded as a regular field vanishing sufficiently fast at spatial infinity. We use (14) and (15) in (46), integrate this equation over a domain of spacetime $\mathfrak{U}$ bounded by two parallel spacelike hyperplanes Σ' and Σ'' with both normals directed toward the future, and a tube T_R of large radius R. Applying the Gauss–Ostrogradskiĭ theorem, we obtain

$$\left(\int_{\Sigma''} - \int_{\Sigma'} + \int_{T_R}\right) \mathrm{d}\sigma_\mu \Theta^{\lambda\mu} + \int_{\tau'}^{\tau''} \mathrm{d}\tau\,(\dot\eta\dot z^\lambda + \eta\ddot z^\lambda) = \frac{1}{4\pi}\int_{\mathfrak{U}} \mathrm{d}^4x\, \mathcal{E}_\mu F^{\lambda\mu} + \int_{\tau'}^{\tau''} \mathrm{d}\tau\, \varepsilon^\lambda. \tag{47}$$

Here, the relation

$$\dot z^\mu \frac{\partial}{\partial x^\mu}\delta^4[x - z(\tau)] = -\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}\frac{\partial}{\partial z^\mu}\delta^4[x - z(\tau)] = -\frac{\mathrm{d}}{\mathrm{d}\tau}\delta^4[x - z(\tau)] \tag{48}$$

has been used to evaluate the integral of $\partial_\mu t^{\lambda\mu}$.

However, our concern here is with the case $\mathcal{E}_\mu = 0$. As soon as the field $F_{\mu\nu}$ is taken to be the retarded solution (36)–(39), equation (47) becomes divergent. To proceed further, a regularization is essential. The singularity must be smeared out over a region bounded by a tube T_ϵ of small radius ϵ enclosing the world line. The regularization scheme may be arbitrary; the only requirement is that it respect the initial symmetries. We can conveniently use a cutoff. The cutoff prescription is to put $F_{\mu\nu} = 0$ within a tube enclosing the world line. To be more specific, we define $\mathrm{Reg}\,\varepsilon^\lambda$ by an appropriate regularization of the left-hand side of (47),

$$\left(\int_{\Sigma''(\epsilon)} - \int_{\Sigma'(\epsilon)} + \int_{T_R}\right) \mathrm{d}\sigma_\mu \Theta^{\lambda\mu} + \int_{\tau'}^{\tau''} \mathrm{d}\tau\,(\dot\eta\dot z^\lambda + \eta\ddot z^\lambda) = \int_{s'}^{s''} \mathrm{d}s\, \mathrm{Reg}\,\varepsilon^\lambda. \tag{49}$$

The cutoff prescription implies perforating the hyperplanes, so that $\Sigma'(\epsilon)$ and $\Sigma''(\epsilon)$ are the perforated hyperplanes with holes of radius ϵ around the points of their intersection with the world line.

We now assume that

$$\mathrm{Reg}\,\varepsilon^\lambda = 0. \tag{50}$$

This gives

$$\left(\int_{\Sigma''(\epsilon)} - \int_{\Sigma'(\epsilon)} + \int_{T_R}\right) \mathrm{d}\sigma_\mu \Theta^{\lambda\mu} + \int_{\tau'}^{\tau''} \mathrm{d}\tau\,(\dot\eta\dot z^\lambda + \eta\ddot z^\lambda) = 0. \tag{51}$$

This equation represents energy–momentum balance of the whole system 'a massless particle plus its field' written in terms of the initial degrees of freedom. We express $\Theta^{\lambda\mu}$ in terms of the retarded solution (36)–(39), and integrate it over $\Sigma'(\epsilon)$ and $\Sigma''(\epsilon)$ to obtain the corresponding 4-momenta of the electromagnetic field.

To ensure Lorentz invariance of this cutoff procedure, we take a hyperplane Σ whose normal is directed along the world line at the point of their intersection. Suppose this hyperplane is intersected by the world line at an instant τ. We define the associated lagging instant $\hat{\tau} = \tau - \epsilon$ with an infinitesimal time lag ϵ, and draw the future light cone C_+ from $z^\mu(\hat{\tau})$. We then delete all points of Σ which fall in the interior of the intersection of Σ and C_+. This gives an invariant (coordinate free) hole on Σ, and renders Σ the desired perforated hyperplane $\Sigma(\epsilon)$. Besides, a truncated future light cone $C_+(\epsilon)$ is manufactured in this way.

The integration surface $\Sigma(\epsilon)$ can be replaced by the surface formed by the truncated future light cone $C_+(\epsilon)$ drawn from the world line at the lagging instant $\hat{\tau}$ and a tube T_R of large radius R enveloping the world line. To see this, we note that the region bounded by $\Sigma(\epsilon)$, $C_+(\epsilon)$, and T_R is free of sources: $\partial_\mu \Theta^{\lambda\mu} = 0$.

A remarkable fact is that the integral over $C_+(\epsilon)$ is completely due to the contribution of the 'near' part of the stress–energy tensor $\Theta_{\rm I}^{\mu\nu}$, containing terms proportional to ρ^{-3} and ρ^{-4}; the 'far' part of the stress–energy tensor $\Theta_{\rm II}^{\mu\nu}$, which goes like ρ^{-2}, yields zero flux through the future light cone. It is demonstrated in appendix A that integrating $\Theta^{\mu\nu}$ over $C_+(\epsilon)$ gives zero. This is just the required result; otherwise we would invoke the renormalization of mass which is problematic in the theory free of dimensional parameters.

Consider the far part of the stress–energy tensor $\Theta_{\rm II}^{\mu\nu}$. By (37), (38), (40) and (44),

$$\Theta_{\rm II}^{\mu\nu} = -\frac{\mathrm{e}^2}{4\pi}\frac{\dddot{z}^2}{\rho^4} R^\mu R^\nu. \tag{52}$$

Since $\Theta_{\rm II}^{\mu\nu}$ behaves like ρ^{-2}, this part of the stress–energy tensor involves integrable singularities near the world line. It remains to manage the ray singularity. A pertinent regularization prescription is to delete the intersection of the integration surface with the singular two-dimensional manifold $\mathcal{M}_2$.

Let us evaluate the 4-momentum associated with $\Theta_{\rm II}^{\mu\nu}$. It is convenient to deform the surface of integration from Σ to the more geometrically motivated surface formed by combining the future light cone with a tubular hypersurface T_ρ enclosing the world line. Indeed, the flux of $\Theta_{\rm II}^{\mu\nu}$ through the future light cone is zero. Furthermore, the area of T_ρ scales as ρ^2 while $\Theta_{\rm II}^{\mu\nu}$ behaves as ρ^{-2}. A tubular surface T_ℓ of a small radius $r = \ell$ enclosing the world line is best suited to our purposes. In a particular Lorentz frame, the surface element is

$$\mathrm{d}\sigma^\mu = n^\mu \ell^2\, \mathrm{d}\Omega\, \mathrm{d}\tau, \qquad n^\mu = (0, \mathbf{n}). \tag{53}$$

Combining (53) with (52) and (26), we obtain

$$\Theta_{\rm II}^{\mu\nu}\, \mathrm{d}\sigma_\nu = -\frac{\mathrm{e}^2\dddot{z}^2}{4\pi}\left(\frac{1}{1-\cos\vartheta}\right)^4 (1, \sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\sin\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\varphi\, \mathrm{d}\tau, \tag{54}$$

and so

$$P_{\rm II}^\mu = \int_{T_\ell} \mathrm{d}\sigma_\nu\, \Theta_{\rm II}^{\mu\nu} = -\frac{2}{3}\mathrm{e}^2\Lambda \int_{-\infty}^{\tau} \mathrm{d}\tau\, \dot{z}^\mu \dddot{z}^2. \tag{55}$$

Here, $\Lambda = 4\delta^{-6}$, with δ being a small number, the lower limit of integration over ϑ, required from the regularization prescription to smear the ray singularity. In the last equation of (55), we have omitted finite terms which are negligibly small in comparison with the term proportional to δ^{-6} in the limit $\delta \to 0$.

If we impose the asymptotic condition

$$\lim_{\tau\to-\infty} \dddot{z}^\mu(\tau) = 0, \tag{56}$$

then the integral over T_R in (51) approaches zero as $R \to \infty$ just as it does in classical electrodynamics of massive charged particles [5].

At first sight, it is possible to interpret $P^\mu_{\rm II}$ as the 4-momentum which is radiated by a charge moving along a null world line. However, close inspection shows that the contribution of $P^\mu_{\rm II}$ to the energy–momentum balance equation can be absorbed by an appropriate reparametrization of the null curve. We use (55) in (51) to make sure the net effect of $P^\mu_{\rm II}$ is gauge removable,

$$\int_{\tau'}^{\tau''} {\rm d}\tau \left(\dot\eta \dot z^\lambda + \eta \ddot z^\lambda - \frac{2}{3}\,{\rm e}^2\Lambda \ddot z^2 \dot z^\lambda\right) = 0. \tag{57}$$

The first and the last terms have similar kinematical structures. This suggests that there is a particular parametrization $\bar\tau$ such that these terms cancel. To verify this suggestion, we go from τ to $\bar\tau$ through the reparametrization[4]

$$d\tau = d\bar\tau \left[1 + \frac{1}{\bar\eta(\bar\tau)}\frac{2}{3}\,{\rm e}^2\Lambda \int_{-\infty}^{\tau} {\rm d}\sigma \ddot z^2(\sigma)\right]. \tag{58}$$

By (8),

$$\eta(\tau) = \bar\eta(\bar\tau) + \frac{2}{3}\,{\rm e}^2\Lambda \int_{-\infty}^{\tau} {\rm d}\sigma\, \ddot z^2(\sigma), \tag{59}$$

and so

$$\dot\eta(\tau) = \dot{\bar\eta}(\bar\tau) + \tfrac{2}{3}\,{\rm e}^2\Lambda \ddot z^2(\tau). \tag{60}$$

Here, the dot denotes the differentiation with respect to τ. If we fix the gauge by imposing the condition $\bar\eta(\bar\tau) = \eta_0$, and take into account that ${\rm d}\bar\tau({\rm d}z^\lambda/{\rm d}\bar\tau) = {\rm d}\tau({\rm d}z^\lambda/{\rm d}\tau)$, then we find that the first and the last terms of (57) cancel out which makes the integrand to be identical to the left-hand side of (9).

This analysis can be extended to a system of several interacting massless particles. Since the general retarded solution to Maxwell's equations is the sum of fields generated by each particle, equations (36)–(39)[5], the stress–energy tensor becomes

$$\Theta^{\mu\nu} = \sum_I \Theta^{\mu\nu}_I + \sum_I \sum_J \Theta^{\mu\nu}_{IJ}, \tag{61}$$

where $\Theta^{\mu\nu}_I$ is comprised of the field $F^{\mu\nu}_I$ due to the Ith charge, and $\Theta^{\mu\nu}_{IJ}$ contains mixed contributions of the fields generated by the Ith and the Jth charges. Following the conventional procedure, we integrate $\Theta^{\mu\nu}_{IJ}$ over a tubular surface T_{ℓ_I} of a small radius ℓ_I enclosing the Ith world line. Without going into detail we simply outline the general idea. The leading singularity, a pole ρ^{-2}, comes from the irregular term $F^{\rm ir}_{\mu\nu}$, while the regular term $F^{\rm r}_{\mu\nu}$ is not sufficiently singular to make a finite contribution to the integral over T_{ℓ_I} in the limit $\ell_I \to 0$. In response to the solid angle integration of ρ^{-2}, the denominator gains the factor $\dot z^2$ which kills the same factor in the numerator of $F^{\rm ir}_{\mu\nu}$, just as it did in establishing (45). The result is

$$\wp^\mu_I = \int_{T_{\ell_I}} {\rm d}\sigma_\nu \sum_J \Theta^{\mu\nu}_{IJ} = -e_I \int_{-\infty}^{\tau_I} {\rm d}\tau_I \sum_J F^{\mu\nu}_{IJ}(z_I)\dot z^I_\nu(\tau_I), \tag{62}$$

where $F^{\mu\nu}_{IJ}(z_I)$ is the retarded field at z_I caused by charge J. We interpret $\wp^\mu_I$ as the 4-momentum extracted from an external field $F^{\mu\nu}_{IJ}(z_I)$ during the whole past history of charge I

[4] In fact, (58) and (8) constitute a set of two functional-differential equations with $\bar\tau = X(\tau)$ and $\bar\eta = Y[\eta(\tau); \bar\tau(\tau)]$ as the unknown functions. It is expected that there exist a positive, regular solutions to these equations.

[5] For simplicity, we omit solutions to the homogeneous field equations describing a free electromagnetic field. If need be, this field could be taken into account in the final result.

prior to the instant τ_I. To put it differently, $\dot{\wp}_I^\mu$ is an external Lorentz force exerted on the Ith particle at z_I.

Let us compare these results with those obtained in the Maxwell–Lorentz theory of massive charged particles. It would be reasonable to begin with the Noether identity (46) in which $T^{\lambda\mu}$, $\mathcal{E}^{\lambda\mu\nu}$, $\mathcal{E}_\mu$ and ε^λ take the same form in both massive and massless cases. For a particle of mass $m \neq 0$, the usual way to explore this identity further is to consider η to be a solution of the constraint equation (20), which implies that the world line is parametrized by the proper time $\mathrm{d}s = \mathrm{d}\tau\sqrt{\dot{z}\cdot\dot{z}}$. Meanwhile there is nothing to prevent us from following the above line. In doing so, we come to equation (51). A closer look at the integrals of $\Theta^{\lambda\mu}$ over $\Sigma'(\epsilon)$ and $\Sigma''(\epsilon)$, representing the 4-momenta of the electromagnetic field at the instants s' and s'', shows, however, a dramatic change of the affair. Indeed, for a particle moving along a timelike world line, we have [14]

$$\left(\int_{\Sigma''(\epsilon)} - \int_{\Sigma'(\epsilon)}\right) \mathrm{d}\sigma_\mu \Theta^{\lambda\mu} = \int_{s'}^{s''} \mathrm{d}s \left(\frac{\mathrm{e}^2}{2\epsilon}\ddot{z}^\lambda - \frac{2}{3}\,\mathrm{e}^2\dddot{z}^\lambda - \frac{2}{3}\,\mathrm{e}^2\ddot{z}^2\dot{z}^\lambda\right). \tag{63}$$

Evidently the term $-\frac{2}{3}\,\mathrm{e}^2\dddot{z}^\lambda$ cannot be canceled by other terms of equation (51), no matter what is the parametrization of the world line. Furthermore, the term $(\mathrm{e}^2/2\epsilon)\ddot{z}^\lambda$ is divergent. For this divergence to be absorbed by the mass renormalization, the gauge must be fixed, $\eta = m/\sqrt{\dot{z}\cdot\dot{z}}$, which implies that the world line is parametrized by the proper time. Accordingly, the term $-\frac{2}{3}\,\mathrm{e}^2\ddot{z}^2\dot{z}^\lambda$ survives in the energy–momentum balance.

To summarize, the energy–momentum balance at a null world line amounts to the equation of motion for a bare particle. The initial degrees of freedom do not experience rearrangement, that is, dressed charged particles and radiation do not arise[6].

5. Direct interparticle action electrodynamics

To claim that massless charged particles do not radiate is another way of stating that there are no unconstrained field degrees of freedom. Every particle is affected by all other particles directly, that is, without mediation of the electromagnetic field. It is therefore tempting to assume that all field degrees of freedom can be integrated out completely without recourse to the Wheeler–Feynman condition of total absorption [15]

$$A^\mu_{\mathrm{ret}}(x) - A^\mu_{\mathrm{adv}}(x) = 0. \tag{64}$$

Naively, this removal of field degrees of freedom can be executed just in equation (13) using the retarded solution (35), with a suitable regularization if required. However, this idea must be abandoned if we are to preserve conformal invariance. The retarded Green's function $D_{\mathrm{ret}}(x) = 2\theta(x_0)\delta(x^2)$ is not conformally invariant due to the presence of the Heaviside step function $\theta(x_0)$. Indeed, a conformal transformation can change the order in which points are lined up along a null ray. We are thus forced to deal with $\bar{D}(x) = \delta(x^2)$, which is specific to the Fokker action involving both retarded and advanced signals. Meanwhile the interaction term of the Fokker action

$$-\frac{1}{2}\sum_I \int \mathrm{d}\tau_I \int \mathrm{d}\tau_J \sum_{J(\neq I)} e_I e_J \dot{z}_I^\mu(\tau_I)\dot{z}_\mu^J(\tau_J)\delta[(z_I - z_J)^2] \tag{65}$$

is devoid of conformal symmetry. To remedy the situation, the Minkowski metric $\eta_{\mu\nu}$ must be substituted for a symmetric tensor of the form [16]

$$h_{\mu\nu}(x-y) = (x-y)^2\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\ln(x-y)^2 = \eta_{\mu\nu} - \frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2}. \tag{66}$$

[6] This is reminiscent of the situation with the Maxwell–Lorentz electrodynamics in a world with one temporal and one spatial dimension in which there is no radiation [4, 5].

Under conformal transformations $d\bar{x}^2 = \sigma^{-2}(x)\,dx^2$, the index μ transforms like a covector at the point x while the index ν transforms like a covector at the point y,

$$\bar{h}_{\mu\nu}(\bar{x} - \bar{y}) = \frac{1}{\sigma(x)\sigma(y)} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{y}^\nu} h_{\alpha\beta}(x - y). \tag{67}$$

Now the action for a conformally invariant action-at-a-distance electrodynamics reads

$$S = -\frac{1}{2}\sum_I^N \int d\tau_I \left\{ \eta_I \dot{z}_I^2 + \int d\tau_J \sum_{J(\neq I)}^N e_I e_J h_{\mu\nu}(z_I - z_J)\dot{z}_I^\mu(\tau_I)\dot{z}_J^\nu(\tau_J)\delta\left[(z_I - z_J)^2\right] \right\}, \tag{68}$$

where the particle sector is chosen to be identical to that of the action (5).

It was shown in [17] that the vector potential adjunct to particle I,

$$A_\mu^J(x) = e_J \int d\tau_J\, h_{\mu\nu}(x - z_J)\dot{z}_J^\nu(\tau_J)\delta[(x - z_J)^2], \tag{69}$$

is an exact solution to Maxwell's equations (10)–(12), in which $F_{\mu\nu}^J = \partial_\mu A_\nu^J - \partial_\nu A_\mu^J$, and all but one of the charges e_I in the current j^μ are assumed to be vanishing. Following the Wheeler and Feynman's original approach [15], one can show that the equation of motion for a massless charged particle (3) in which $F^{\mu\nu}$ is the retarded field adjunct to all other charges derives from (68) provided that equation (64) holds.

We thus see that the Wheeler–Feynman condition of total absorption (64) remains essential for the action-at-a-distance electrodynamics of massless charged particles. Recall that there are two alternative concepts of radiation, proposed by Dirac and Teitelboim (for a review see [2]). Although these concepts have some points in common, they are not equivalent. Accordingly, (64) does not amount to the lack of radiation in the sense of Teitelboim whose definition was entertained in the previous section.

6. Massless quarks

Consider massless colored particles, or simply massless quarks[7]. The Lorentz force law is changed for the Wong force law [18],

$$\eta\ddot{z}^\mu + \dot{\eta}\dot{z}^\mu = \dot{z}_\nu\, \mathrm{tr}(QG^{\mu\nu}). \tag{70}$$

In other words, a particle carrying color charge $Q = Q^a T_a$ is affected by the Yang–Mills field $G_{\mu\nu} = G_{\mu\nu}^a T_a$ at the point of its location z^μ, as indicated by (70). We begin with the case of a single quark, and adopt the simplest non-Abelian gauge group $SU(2)$.

The color charge is governed by

$$\dot{Q} = -\mathrm{i}g[Q, \dot{z}^\mu A_\mu], \tag{71}$$

where g is the Yang–Mills coupling constant. The Yang–Mills equations read

$$D_\mu G^{\mu\nu} = 4\pi j^\nu. \tag{72}$$

Here, $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative, and j^μ is the color current,

$$j^\mu(x) = \int_{-\infty}^{\infty} d\tau\; Q(\tau)\dot{z}^\mu(\tau)\delta^4[x - z(\tau)]. \tag{73}$$

[7] For a systematic study of a classical theory of particles interacting with non-Abelian gauge fields, the so-called Yang–Mills–Wong theory, see [3, 5].

Green's function technique does not apply to the nonlinear partial differential equations (72)–(73). We, therefore, impose the retarded condition for the Yang–Mills field propagation, and furnish the ansatz

$$A^{\mu} = \sum_{a=1}^{3} T_a(\tau)(\dot{z}^{\mu}\Phi^a + R^{\mu}\Psi^a). \tag{74}$$

Retracing essential steps in the Yang–Mills–Wong theory of massive quarks [3, 5], with appropriate modifications, we find that the retarded solution to equations (71)–(73) is given by[8]

$$A^{\mu} = \mathcal{Q}\frac{\dot{z}^{\mu}}{\rho}. \tag{75}$$

Here, $\mathcal{Q} = T_a\mathcal{Q}^a$, $\mathcal{Q}^a$ are arbitrary integration constants. This solution is unique, modulo gauge terms proportional to $\mathcal{Q}R^{\mu}/\rho = \mathcal{Q}\partial^{\mu}\tau$. Equation (75) describes an Abelian field.

Anticipating that an irregular term of the field strength, similar to that defined in (39), is responsible for the Gauss' surface-integration procedure, we identify $\mathcal{Q}$ with the color charge Q appearing in (73). Because equation (72) is covariant under the gauge transformations

$$A^{\mu} \to \Omega\left(A^{\mu} + \frac{\mathrm{i}}{g}\partial_{\mu}\right)\Omega^{\dagger}, \qquad j_{\mu} \to \Omega j_{\mu}\Omega^{\dagger}, \tag{76}$$

one can find a unitary matrix Ω to diagonalize the Hermitian matrix j_{μ}. Accordingly, the vector potential (75) is transformed to the form involving only commuting matrices which span the Cartan subalgebra. In this case, that is, for the gauge group $SU(2)$, if the color basis elements T_a are expressed in terms of the Pauli matrices $T_a = \frac{1}{2}\sigma_a$, the diagonalized color charge is $Q = \frac{1}{2}\sigma_3 Q^3$.

The regular part of the gluon field strength is given by

$$G^{\mu\nu} = R^{\mu}W^{\nu} - R^{\nu}W^{\mu}, \tag{77}$$

where

$$W^{\mu} = \frac{Q}{\rho^2}\left(-\dot{z}^{\mu}\frac{\ddot{z}\cdot R}{\rho} + \ddot{z}^{\mu}\right). \tag{78}$$

Thus, in $\overline{\mathbb{M}}_4$, a massless quark generates an Abelian field whose regular part is a null field, ${}^*G_{\mu\nu}G^{\mu\nu} = 0$, $G_{\mu\nu}G^{\mu\nu} = 0$.

By repeating what was done in section 4, we find that a massless quark does not radiate.

This consideration can be extended to the case of N massless quarks and the unitary group $SU(\mathcal{N})$ with arbitrary N and $\mathcal{N} \geqslant 2$. The vector potential

$$A^{\mu} = \sum_{I=1}^{N} Q_I\frac{\dot{z}_I^{\mu}}{\rho_I} \tag{79}$$

represents the generic retarded Abelian solution to the Yang–Mills equations. Here,

$$Q_I = \sum_{a=1}^{\mathcal{N}-1} e_I^a H_a, \tag{80}$$

where e_I^a are arbitrary real coefficients. The generators H_a belong to the Cartan subalgebra of the Lie algebra $su(\mathcal{N})$.

Similar to classical electrodynamics of massless charged particles, the Yang–Mills–Wong theory of massless quarks is invariant under $C(1, 3)$, and hence defies its formulation as an ordinary Lagrangian system in $\mathbb{R}_{1,3}$.

[8] The procedure of finding Φ^a and Ψ^a resembles that performed in section 3 in many ways.

7. Discussion and outlook

For comparison, we briefly review the Yang–Mills–Wong theory of massive quarks.

The field sector of this theory is invariant under $C(1,3)$, but this invariance is violated in the particle sector. There are two classes of retarded solutions A_μ to the Yang–Mills equations, non-Abelian and Abelian [3, 5]. To be specific, we refer to the case of N massive quarks and the gauge group $SU(\mathcal{N})$, with arbitrary N such that $\mathcal{N} \geqslant N+1$. The gauge group of non-Abelian solutions is spontaneously deformed to $SL(\mathcal{N},\mathbb{R})$. These solutions represent gauge fields of magnetic type. These solutions involve terms which are explicitly conformally non-invariant. An accelerated quark gains (rather than loses) energy by emitting the Yang–Mills field of this type. A plausible interpretation of the solutions with deformed gauge invariance is that such configurations describe Bose condensates of gluon fields in the hadronic phase. By contrast, the gauge group of Abelian solutions is $SU(\mathcal{N})$. These solutions are conformally invariant constructions. They represent gauge fields of electric type. An accelerated quark loses energy by emitting the Yang–Mills field of this type. These Abelian solutions are associated with the QGP vacuum.

The Yang–Mills–Wong theory of massless quarks is perfectly invariant under $C(1,3)$. There are only Abelian solutions to the Yang–Mills equations, equation (79). This is because only such constructions are compatible with the conformal symmetry requirement. The regular part of the field, equations (77)–(78), represents a null-field configuration. Accelerated quarks neither gain nor lose energy by emitting this Yang–Mills field. It is natural to think of such solutions as Bose condensates of gluon fields in QGP.

Conceivably the Yang–Mills–Wong theory of massless scalar quarks leaves room for the direct action formulation

$$S=-\frac{1}{2}\sum_{I=1}^{N}\int d\tau_I\left\{\eta_I\dot{z}_I^2+\sum_{J=1}^{N}\mathrm{tr}(Q_IQ_J)\int d\tau_J h_{\mu\nu}(z_I-z_J)\dot{z}_I^{\mu}(\tau_I)\dot{z}_J^{\nu}(\tau_J)\delta[(z_I-z_J)^2]\right\}. \tag{81}$$

This Fokker-type action results from the fact that the Yang–Mills sector is linearized (that is, becomes essentially the same as the Maxwell sector) when the color dynamics is confined to the Cartan subgroup. Maintaining the color dynamics in this Abelian regime is controlled by conformal invariance.

Equation (81) can form the basis of a first-quantized path integral description of this system. It has long been known [19–21] that, for all processes in scalar QED in which the total number of real photons is zero, the conventional current–field interaction used in the S matrix for a collection of species of particles, given by

$$\sum_I\int d^4x\, j_I^{\mu}(x)A_\mu(x), \tag{82}$$

may be replaced by the direct current–current interaction given by

$$\frac{1}{2}\sum_I\sum_J\int d^4x\, d^4y\, j_{I\mu}(x)D_{\mathrm{F}}(x-y)j_J^{\mu}(y) \tag{83}$$

without change in the results. Here, $D_{\mathrm{F}}(x)$ is the Feynman propagator. It is related to the Fokker propagator $\bar{D}(x)=\delta(x^2)$ by

$$D_{\mathrm{F}}=\bar{D}+\tfrac{1}{2}(D^{+}-D^{-}), \tag{84}$$

where D^{+} is the positive frequency part of the Pauli–Jordan function $D = D_{\rm ret} - D_{\rm adv}$. With this decomposition, we have two sets of terms. The first set

$$\frac{1}{2}\sum_{I}\sum_{J}\int \mathrm{d}^4x\,\mathrm{d}^4y\, j_{I\mu}(x)\bar{D}(x-y)j_J^{\mu}(y) \tag{85}$$

will be recognized as the conventional Fokker coupling between the charged currents. The second set of terms can be brought into the form

$$\frac{1}{2}\sum_{I}\sum_{J}\int \mathrm{d}^4x\,\mathrm{d}^4y\, j_{I\mu}(x)D^{+}(x-y)j_J^{\mu}(y). \tag{86}$$

This expression must in some way represent the response of the universe. For a system enclosed in a light tight box, (86) does not contribute to the S matrix [20], and, therefore, (83) and (85) give the same results.

Turning to massless quarks, similar reasoning shows that the contribution of

$$\frac{1}{2}\sum_{I}\sum_{J}\mathrm{tr}(Q_IQ_J)\int \mathrm{d}\tau_I\int \mathrm{d}\tau_J\, h_{\mu\nu}(z_I - z_J)\dot{z}_I^{\mu}(\tau_I)\dot{z}_J^{\nu}(\tau_J)D^{+}(z_I - z_J) \tag{87}$$

to the S matrix in quantum chromodynamics must vanish. Now a QGP lump plays the same role as the light tight box in the Wheeler–Feynman electrodynamics. Hence, substituting the Fokker propagator $\bar{D}$ by the Feynman propagator $D_{\rm F}$ in (81) will be of no consequences.

With $D_{\rm F}$ in place of $\bar{D}$, one may perform the Wick rotation. Then all world lines in the path integral become curves in Euclidean spacetime $\mathbb{R}_4$. The conformal group acting on this arena is $C(4)$. The only remnant of the initial conformal structure in this Euclideanized picture is the conformal metric $h_{\mu\nu}$ defined in (66). A consistent Euclidean direct action formulation reads[9]

$$S_{\rm E} = \frac{1}{2}\sum_{I=1}^{N}\int \mathrm{d}\tau_I\left\{\eta_I\dot{z}_I^2 + \sum_{J=1}^{N}\mathrm{tr}(Q_IQ_J)\int \mathrm{d}\tau_J\,\dot{z}_I^{\mu}(\tau_I)\frac{h_{\mu\nu}(z_I - z_J)}{(z_I - z_J)^2}\dot{z}_J^{\nu}(\tau_J)\right\}. \tag{88}$$

A more realistic model of QGP arises if quark spin is taken into account. For this purpose we can conveniently follow the much-studied procedure [22].

To sum up, we have shown that, when moving with acceleration, charged and colored zero-mass particles do not radiate. More generally, classical electrodynamics of massless charged particles and the Yang–Mills–Wong theory of massless quarks do not experience rearranging their initial degrees of freedom into dressed particles and radiation. We have found that a conformally invariant version of the direct interparticle action theory can be formulated, in the manner of Wheeler and Feynman, for both charged and colored massless particles.

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[9] *Note added in proof.* Strickly speaking, the action (88) represents a consistent Euclidean dynamics only in the context of the Feynman's path integral approach. Indeed, turning to the principle of least action, we come to equation (1) which implies that every world line $Z_I^{\mu}(\tau_I)$ in $\mathbb{R}_4$ is actually reduced to a point, and hence the classical history of this Euclideanized system is trivial.

Appendix A

In this appendix, we show that the regular part of the electromagnetic field generated by a massless charged particle $F^{\mathrm{r}}_{\mu\nu} = R_\mu V_\nu - R_\nu V_\mu$ does not contribute to the flux through a surface enclosing the charge. We omit the label 'r', and consider the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in a particular Lorentz frame in which

$$\dot{z}^\mu = (1, \mathbf{v}), \qquad \ddot{z}^\mu = (0, \mathbf{a}), \qquad R^\mu = r(1, \mathbf{n}), \qquad \mathbf{n}^2 = 1. \tag{A.1}$$

By (1) and (2),

$$\mathbf{v}^2 = 1, \qquad \mathbf{v}\cdot\mathbf{a} = 0. \tag{A.2}$$

Using (A.1), we write

$$\rho = R\cdot v = r(1 - \mathbf{n}\cdot\mathbf{v}), \qquad R\cdot\ddot{z} = -r(\mathbf{n}\cdot\mathbf{a}), \tag{A.3}$$

and

$$V_\mu = \frac{q}{r^2(1-\mathbf{n}\cdot\mathbf{v})^2}\left(\frac{\mathbf{n}\cdot\mathbf{a}}{1-\mathbf{n}\cdot\mathbf{v}}, -\mathbf{v}\frac{\mathbf{n}\cdot\mathbf{a}}{1-\mathbf{n}\cdot\mathbf{v}} - \mathbf{a}\right). \tag{A.4}$$

Therefore, the electric field $\mathbf{E}_i = F_{0i} = R_0 V_i - R_i V_0$ is

$$\mathbf{E} = \frac{q}{r(1-\mathbf{n}\cdot\mathbf{v})^2}\left[(\mathbf{n}-\mathbf{v})\frac{\mathbf{n}\cdot\mathbf{a}}{1-\mathbf{n}\cdot\mathbf{v}} - \mathbf{a}\right]. \tag{A.5}$$

It is clear that $\mathbf{E}$ is regular for any direction, except for $\mathbf{n} = \mathbf{v}$, and that $\mathbf{E}\cdot\mathbf{n} = 0$.

Likewise, the magnetic field $\mathbf{B}_i = -\frac{1}{2}\epsilon_{ijk}F^{jk} = \epsilon_{ijk}V^j R^k$ is

$$\mathbf{B} = \frac{q}{r(1-\mathbf{n}\cdot\mathbf{v})^2}\mathbf{n}\times\left(\mathbf{v}\frac{\mathbf{n}\cdot\mathbf{a}}{1-\mathbf{n}\cdot\mathbf{v}} + \mathbf{a}\right), \tag{A.6}$$

whence it follows that $\mathbf{B}\cdot\mathbf{n} = 0$.

From (A.5) and (A.6) it will be noted that the electric and magnetic fields are of the same strength,

$$|\mathbf{E}| = |\mathbf{B}| = \frac{q|\mathbf{a}|}{r(1-\mathbf{n}\cdot\mathbf{v})^2}, \tag{A.7}$$

and perpendicular to each other, as might be expected. We thus have a triplet of mutually orthogonal vectors $\mathbf{E}, \mathbf{B}$ and $\mathbf{n}$. Since $\mathbf{n}$ is normal to the surface enclosing the charge, the infinitesimal fluxes of $\mathbf{E}$ and $\mathbf{B}$ through the appropriate surface element are vanishing.

It remains to see whether the fluxes of $\mathbf{E}$ and $\mathbf{B}$ through a surface enclosing the singular ray along $\mathbf{v}$ are zero. Let the charge be located at the origin, and $\mathbf{v}$ be parallel to the z-axis. We take a tube T_ϵ of small radius ϵ enclosing the singular ray, and denote its normal by $\mathbf{u}$. We attach a hemisphere S_ϵ of radius ϵ, centered at the origin, to the tube T_ϵ. A point $\mathbf{x}$ on T_ϵ is separated from the origin by $r = \sqrt{z^2+\epsilon^2}$. The unit vector $\mathbf{n}$ directed to $\mathbf{x}$ is represented as

$$\mathbf{n} = \frac{1}{r}(z\mathbf{v} + \epsilon\mathbf{u}), \tag{A.8}$$

so that

$$\mathbf{n}\cdot\mathbf{a} = \frac{\epsilon}{r}(\mathbf{u}\cdot\mathbf{a}). \tag{A.9}$$

Introducing zenith and azimuth angles ϑ and φ, we obtain $\mathbf{u}\cdot\mathbf{a} = a\sin\vartheta\cos\varphi$.

With this preliminary,

$$\mathbf{E}\cdot\mathbf{u} = \frac{qz}{(r-z)^2}(\mathbf{u}\cdot\mathbf{a}) = \frac{qaz}{(r-z)^2}\sin\vartheta\cos\varphi. \tag{A.10}$$

Integrating (A.10) over φ from 0 to 2π, one finds that the total flux of $\mathbf{E}$ through T_ϵ equals zero. It is interesting that there are both flux flowing inward the tube and flux directed outward from it, which exactly cancel. The flux of $\mathbf{E}$ through the hemisphere S_ϵ is zero because $\mathbf{E}\cdot\mathbf{n} = 0$. The flux through the cross section of the tube T_ϵ disappears in the limit $r \to \infty$ due to the suppressing factor r^{-1}. However, we must handle this divergent integral with caution. We use polar coordinates ϱ and φ so that $r^2 = \varrho^2 + z^2$, and introduce a cutoff parameter δ to bound the integration over ϱ within the limits $\epsilon \geqslant \varrho \geqslant \delta$. We then complete the definition of this surface integral by letting the parameter z to go to ∞ before removing the cutoff $\delta \to 0$. This makes it clear that the flux of $\mathbf{E}$ through the cross section of T_ϵ is indeed vanishing.

The same statement holds for the total flux of $\mathbf{B}$.

Appendix B

In this appendix, we show that integrating the stress–energy tensor over the future light cone C_+ drawn from a point on the world line gives zero. We first consider the term $\Theta^{\rm ir}_{\mu\nu}$ built from $F^{\rm ir}_{\mu\nu}$. By (39),

$$F^{\rm ir}_{\mu}{}^{\alpha}F^{\rm ir}_{\alpha\nu} + \frac{1}{4}\eta_{\mu\nu}F^{\rm ir}_{\alpha\beta}F^{\rm ir\,\alpha\beta} = {\rm e}^2\frac{(\dot z^2)^2}{\rho^4}\left(c_\mu\dot z_\nu + c_\nu\dot z_\mu - \dot z^2 c_\mu c_\nu - \frac{1}{2}\eta_{\mu\nu}\right). \tag{B.1}$$

With the surface element on C_+,

$$d\sigma^\mu = c^\mu\rho^2\,d\rho\,d\Omega, \tag{B.2}$$

we have

$$\Theta^{\rm ir}_{\mu\nu}\,d\sigma^\nu = {\rm e}^2\frac{(\dot z^2)^2}{8\pi\rho^2}c_\mu. \tag{B.3}$$

We retain only the term of the second order in $\dot z^2$, and drop higher order terms.

To define the corresponding 4-momentum of the electromagnetic field $P^{\rm ir}_\mu$, we must introduce a regularization. A convenient coordinate-free regularization is a cutoff that renders the future light cone C_+ with vertex at $z^\mu(\hat\tau)$ a truncated light cone $C_+(\epsilon)$ whose truncation surface arises from the intersection of this light cone with a hyperplane Σ perpendicular to the world line $z^\mu(\tau)$ at the instant τ the world line goes through Σ. In response to the solid angle integration of ρ^{-2}, the denominator gains the factor $\dot z^2$ just as it did in establishing (45). However, this factor cannot kill the factor $(\dot z^2)^2$ in the numerator, and hence the cutoff-regularized 4-momentum, involving the overall zero factor $\dot z^2$, is vanishing. In the limit of cutoff removal $\epsilon \to 0$, we have $P^{\rm ir}_\mu = 0$.

Consider the term of stress–energy tensor containing mixed contribution of $F^{\rm r}_{\mu\nu}$ and $F^{\rm ir}_{\mu\nu}$. By (37)–(40) and (43),

$$F^{\rm ir}_{\mu}{}^{\alpha}F^{\rm r}_{\alpha\nu} + F^{\rm r}_{\mu}{}^{\alpha}F^{\rm ir}_{\alpha\nu} = {\rm e}^2\frac{\dot z^2}{\rho^3}[(c_\mu V_\nu + c_\nu V_\mu) - 2c_\mu c_\nu(\dot z\cdot V)], \tag{B.4}$$

and so

$$F^{\rm ir\,\alpha\beta}F^{\rm r}_{\alpha\beta} = 0. \tag{B.5}$$

Contracting (B.4) with the surface element $d\sigma^\nu$, defined in (B.2), gives

$$\left(F^{\rm ir}_{\mu}{}^{\alpha}F^{\rm r}_{\alpha\nu} + F^{\rm r}_{\mu}{}^{\alpha}F^{\rm ir}_{\alpha\nu}\right)d\sigma^\nu = 0. \tag{B.6}$$

This completes proof of our assertion.

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